Poisson Processes
Poisson Random Measures

Étienne Pardoux

Aix Marseille Université

MoCLiS, Dschang, Nov. 24, 2025

The Poisson Process

- Recall that for $a > 0$, a r.v. $X \sim Poi(a)$ if $\mathbb{P}(X = k) = \frac{a^k}{k!} e^{-a}$. Then $\mathbb{E}[X] = a$, $\mathrm{Var}(X) = a$, and for $s \in \mathbb{R}$, $\mathbb{E}s^X = e^{a(s-1)}$.
- Given $\lambda > 0$, a rate λ Poisson process $\{P_t,\ t \geq 0\}$ is an $\mathbb{N}$-valued random process whose increments on disjoint intervals are independent, and for $0 \leq s < t$, $P_t - P_s \sim Poi(\lambda(t-s))$.
- This means that P_t counts a Poissonian random number of points on the interval $[0, t]$.

The Poisson Process

- Recall that for $a > 0$, a r.v. $X \sim Poi(a)$ if $\mathbb{P}(X = k) = \frac{a^k}{k!}e^{-a}$. Then $\mathbb{E}[X] = a$, $\mathrm{Var}(X) = a$, and for $s \in \mathbb{R}$, $\mathbb{E}s^X = e^{a(s-1)}$.
- Given $\lambda > 0$, a rate λ Poisson process $\{P_t,\ t \geq 0\}$ is an $\mathbb{N}$-valued random process whose increments on disjoint intervals are independent, and for $0 \leq s < t$, $P_t - P_s \sim Poi(\lambda(t-s))$.
- This means that P_t counts a Poissonian random number of points on the interval $[0, t]$.

The Poisson Process

- Recall that for $a > 0$, a r.v. $X \sim Poi(a)$ if $\mathbb{P}(X = k) = \frac{a^k}{k!} e^{-a}$. Then $\mathbb{E}[X] = a$, $\text{Var}(X) = a$, and for $s \in \mathbb{R}$, $\mathbb{E}s^X = e^{a(s-1)}$.
- Given $\lambda > 0$, a rate λ Poisson process $\{P_t,\ t \geq 0\}$ is an $\mathbb{N}$-valued random process whose increments on disjoint intervals are independent, and for $0 \leq s < t$, $P_t - P_s \sim Poi(\lambda(t - s))$.
- This means that P_t counts a Poissonian random number of points on the interval $[0, t]$.

Poisson random measures

- I will consider only *standard* Poisson random measures on $\mathbb{R}_+^2$.
- A (standard) Poisson random measure Q on $\mathbb{R}_+^2$ is a sum of Dirac measures at random points in $\mathbb{R}_+^2$, which is such that
  1. For any Borel subset $A \subset \mathbb{R}_+^2$, $Q(A) \sim Poi(Leb(A))$.
  2. For any disjoint subsets $A_1, A_2, \ldots, A_n \subset \mathbb{R}_+^2$, the r.v.s $Q(A_1), Q(A_2), \ldots, Q(A_n)$ are independent.
- For any $\lambda > 0$,

$$\left\{\int_0^t \int_0^\infty 1_{u \le \lambda} Q(ds, du) = Q([0,t] \times [0,\lambda]),\ t \ge 0\right\}$$

is a rate λ Poisson process.

- If $\varphi : \mathbb{R}_+ \mapsto \mathbb{R}_+$ is measurable and locally integrable on $\mathbb{R}_+$,

$$\int_0^t \int_0^\infty 1_{u \le \varphi(s)} Q(ds, du) = \int_0^t \int_0^{\varphi(s)} Q(ds, du) \sim Poi\left(\int_0^t \varphi(s) ds\right).$$

Poisson random measures

- I will consider only *standard* Poisson random measures on $\mathbb{R}_+^2$.
- A (standard) Poisson random measure Q on $\mathbb{R}_+^2$ is a sum of Dirac measures at random points in $\mathbb{R}_+^2$, which is such that
  1. For any Borel subset $A \subset \mathbb{R}_+^2$, $Q(A) \sim Poi(Leb(A))$.
  2. For any disjoint subsets $A_1, A_2, \ldots, A_n \subset \mathbb{R}_+^2$, the r.v.s $Q(A_1), Q(A_2), \ldots, Q(A_n)$ are independent.
- For any $\lambda > 0$,

$$\left\{ \int_0^t \int_0^\infty 1_{u \le \lambda} Q(ds, du) = Q([0, t] \times [0, \lambda]), \ t \ge 0 \right\}$$

is a rate λ Poisson process.

- If $\varphi : \mathbb{R}_+ \mapsto \mathbb{R}_+$ is measurable and locally integrable on $\mathbb{R}_+$,

$$\int_0^t \int_0^\infty 1_{u \le \varphi(s)} Q(ds, du) = \int_0^t \int_0^{\varphi(s)} Q(ds, du) \sim Poi\left(\int_0^t \varphi(s) ds \right).$$

Poisson random measures

- I will consider only *standard* Poisson random measures on $\mathbb{R}_+^2$.
- A (standard) Poisson random measure Q on $\mathbb{R}_+^2$ is a sum of Dirac measures at random points in $\mathbb{R}_+^2$, which is such that
  1. For any Borel subset $A \subset \mathbb{R}_+^2$, $Q(A) \sim Poi(Leb(A))$.
  2. For any disjoint subsets $A_1, A_2, \ldots, A_n \subset \mathbb{R}_+^2$, the r.v.s $Q(A_1), Q(A_2), \ldots, Q(A_n)$ are independent.
- For any $\lambda > 0$,

$$\left\{ \int_0^t \int_0^\infty 1_{u \le \lambda} Q(ds, du) = Q([0,t] \times [0,\lambda]),\ t \ge 0 \right\}$$

  is a rate λ Poisson process.
- If $\varphi : \mathbb{R}_+ \mapsto \mathbb{R}_+$ is measurable and locally integrable on $\mathbb{R}_+$,

$$\int_0^t \int_0^\infty 1_{u \le \varphi(s)} Q(ds, du) = \int_0^t \int_0^{\varphi(s)} Q(ds, du) \sim Poi\left(\int_0^t \varphi(s) ds \right).$$

Poisson random measures

- I will consider only *standard* Poisson random measures on $\mathbb{R}_+^2$.
- A (standard) Poisson random measure Q on $\mathbb{R}_+^2$ is a sum of Dirac measures at random points in $\mathbb{R}_+^2$, which is such that
  1. For any Borel subset $A \subset \mathbb{R}_+^2$, $Q(A) \sim Poi(Leb(A))$.
  2. For any disjoint subsets $A_1, A_2, \ldots, A_n \subset \mathbb{R}_+^2$, the r.v.s $Q(A_1), Q(A_2), \ldots, Q(A_n)$ are independent.
- For any $\lambda > 0$,

$$\left\{ \int_0^t \int_0^\infty 1_{u \le \lambda} Q(ds, du) = Q([0,t] \times [0,\lambda]),\ t \ge 0 \right\}$$

  is a rate λ Poisson process.
- If $\varphi : \mathbb{R}_+ \mapsto \mathbb{R}_+$ is measurable and locally integrable on $\mathbb{R}_+$,

$$\int_0^t \int_0^\infty 1_{u \le \varphi(s)} Q(ds, du) = \int_0^t \int_0^{\varphi(s)} Q(ds, du) \sim Poi\left(\int_0^t \varphi(s) ds \right).$$

A martingale

- We define the compensated measure $\bar{Q}(ds, du) = Q(ds, du) - ds\, du$. Then
$$M_t = \int_0^t \int_0^{\varphi(s)} \bar{Q}(ds, du)$$
is a process with zero mean independent increments, hence a martingale. With $< M >_t := \int_0^t \varphi(s)ds$, $M_t^2 - < M >_t$ is also a martingale, hence $\mathbb{E}[M_t^2] = \int_0^t \varphi(s)ds$.
- Note that the mean and the variance of $\int_0^t \int_0^{\varphi(s)} Q(ds, du)$ are equal, since it is a Poisson r.v.

A martingale

- We define the compensated measure $\bar{Q}(ds, du) = Q(ds, du) - ds\, du$. Then
$$M_t = \int_0^t \int_0^{\varphi(s)} \bar{Q}(ds, du)$$
is a process with zero mean independent increments, hence a martingale. With $< M >_t := \int_0^t \varphi(s) ds$, $M_t^2 - < M >_t$ is also a martingale, hence $\mathbb{E}[M_t^2] = \int_0^t \varphi(s) ds$.
- Note that the mean and the variance of $\int_0^t \int_0^{\varphi(s)} Q(ds, du)$ are equal, since it is a Poisson r.v.

Generalization

- We want now to consider the same objects, but with φ random. If φ and Q are independent, provided $\mathbb{E}\int_0^T \varphi(s)ds < \infty$, $\forall T > 0$, again M_t and $M_t^2 - <M>_t$ are martingales.
- The same is still true without the independence assumption, provided that $\varphi(t)$ and $Q|_{[t,+\infty)\times\mathbb{R}_+}$ are independent for all t.
  The proper assumption is that $\varphi(t)$ should be predictable w.r.t. a filtration $\mathcal{F}_t$ such that $Q|_{[0,t]\times\mathbb{R}_+}$ is $\mathcal{F}_t$-measurable, $\forall t \geq 0$.
- In what follows, $\varphi(t)$ will be independent of $Q|_{(t,+\infty)\times\mathbb{R}_+}$, not of $Q|_{[t,+\infty)\times\mathbb{R}_+}$. For that reason, we shall consider rather the process $\int_0^t \int_0^{\varphi(s^-)} Q(ds, du)$. Hence the martingales will be

$$M_t = \int_0^t \int_0^{\varphi(s^-)} \bar{Q}(ds, du), \quad <M>_t = \int_0^t \varphi(s)ds\,.$$

  Hence in particular $\mathbb{E}[M_t^2] = \mathbb{E}\int_0^t \varphi(s)ds$.

Generalization

- We want now to consider the same objects, but with φ random. If φ and Q are independent, provided $\mathbb{E}\int_0^T \varphi(s)ds < \infty$, $\forall T > 0$, again M_t and $M_t^2 - < M >_t$ are martingales.
- The same is still true without the independence assumption, provided that $\varphi(t)$ and $Q|_{[t,+\infty)\times\mathbb{R}_+}$ are independent for all t.
  The proper assumption is that $\varphi(t)$ should be predictable w.r.t. a filtration $\mathcal{F}_t$ such that $Q|_{[0,t]\times\mathbb{R}_+}$ is $\mathcal{F}_t$-measurable, $\forall t \geq 0$.
- In what follows, $\varphi(t)$ will be independent of $Q|_{(t,+\infty)\times\mathbb{R}_+}$, not of $Q|_{[t,+\infty)\times\mathbb{R}_+}$. For that reason, we shall consider rather the process $\int_0^t \int_0^{\varphi(s^-)} Q(ds, du)$. Hence the martingales will be

$$M_t = \int_0^t \int_0^{\varphi(s^-)} \bar{Q}(ds, du), \quad < M >_t = \int_0^t \varphi(s)ds\,.$$

Hence in particular $\mathbb{E}[M_t^2] = \mathbb{E}\int_0^t \varphi(s)ds$.

Generalization

- We want now to consider the same objects, but with φ random. If φ and Q are independent, provided $\mathbb{E}\int_0^T \varphi(s)ds < \infty$, $\forall T > 0$, again M_t and $M_t^2 - < M >_t$ are martingales.
- The same is still true without the independence assumption, provided that $\varphi(t)$ and $Q|_{[t,+\infty)\times\mathbb{R}_+}$ are independent for all t.
The proper assumption is that $\varphi(t)$ should be predictable w.r.t. a filtration $\mathcal{F}_t$ such that $Q|_{[0,t]\times\mathbb{R}_+}$ is $\mathcal{F}_t$-measurable, $\forall t \geq 0$.
- In what follows, $\varphi(t)$ will be independent of $Q|_{(t,+\infty)\times\mathbb{R}_+}$, not of $Q|_{[t,+\infty)\times\mathbb{R}_+}$. For that reason, we shall consider rather the process $\int_0^t \int_0^{\varphi(s^-)} Q(ds, du)$. Hence the martingales will be

$$M_t = \int_0^t \int_0^{\varphi(s^-)} \bar{Q}(ds, du), \quad < M >_t = \int_0^t \varphi(s)ds\,.$$

Hence in particular $\mathbb{E}[M_t^2] = \mathbb{E}\int_0^t \varphi(s)ds$.