Law of Large Numbers for the SIR model

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The case of the Markovian model 1

- Here we assume that each infected individual infects at rate λ during a period of length $\mathcal{I}_\rangle \sim Exp(\mu)$, where the $\mathcal{I}_\rangle$'s are i.i.d..
- The model for the evolution of $(S^N(t), I^N(t), R^N(t))$ is the following

$$S^N(t) = S^N(0) - \int_0^t \int_0^{\lambda \bar{S}^N(s^-) I^N(s^-)} Q_{inf}(ds, du),$$

$$I^N(t) = I^N(0) + \int_0^t \int_0^{\lambda \bar{S}^N(s^-) I^N(s^-)} Q_{inf}(ds, du) - \int_0^t \int_0^{\mu I^N(s^-)} Q_{rec}(ds, du),$$

$$R^N(t) = R^N(0) + \int_0^t \int_0^{\mu I^N(s^-)} Q_{rec}(ds, du).$$

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The case of the Markovian model 2

- We divide the above system by N, yielding

$$\bar{S}^N(t) = S^N(0) - \lambda \int_0^t \bar{S}^N(s)\bar{I}^N(s)ds - \varepsilon^N_{inf}(t),$$

$$\bar{I}^N(t) = I^N(0) + \lambda \int_0^t \bar{S}^N(s)\bar{I}^N(s)ds + \varepsilon^N_{inf}(t) - \mu \int_0^t I^N(s)ds - \varepsilon^N_{rec}(t)$$

$$\bar{R}^N(t) = R^N(0) + \mu \int_0^t I^N(s)ds + \varepsilon^N_{rec}(t),$$

- where

$$\varepsilon^N_{inf}(t) = \frac{1}{N}\int_0^t \int_0^{\lambda \bar{S}^N(s^-)I^N(s^-)} \bar{Q}_{inf}(ds, du),$$

$$\varepsilon^N_{rec}(t) = \frac{1}{N}\int_0^t \int_0^{\mu I^N(s^-)} \bar{Q}_{rec}(ds, du)\,.$$

- Clearly $\mathbb{E}[|\varepsilon^N_{inf}(t)|^2] \le \frac{\lambda^2}{N}\int_0^t \bar{S}^N(s)\bar{I}^N(s)ds \le \frac{\lambda^2 t}{N}$, $\mathbb{E}[|\varepsilon^N_{rec}(t)|^2] \le \frac{t}{N}$.

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The VI model 1

- We now write the VI N-model. First we are given $(S^N(0), I^N(0), R^N(0))$, such that $S^N(0) + I^N(0) + R^N(0) = N$.
- To the initially infected individuals, we associate $\{\lambda_j^0(t),\ t \geq 0,\ 1 \leq j \leq I^N(0)\}$, and define $\eta_j^0 = \sup\{t, \lambda_j^0(t) > 0\}$.
- Initially susceptible individuals get infected at times τ_i^N, $1 \leq i \leq S^N(0)$. To each i, we associate $\lambda_i(t)$, which is the infectivity at time t after its infection time. Hence at time t, the infectivity of individual i is $\lambda_j(t - \tau_i^N)$. We assume that $\lambda_i(t) = 0$ for $t < 0$. We define $\eta_i = \sup\{t > 0,\ \lambda_i(t) > 0\}$.
- We assume that the two sequences $\{\lambda_j^0(\cdot),\ 1 \leq j \leq I^N(0)\}$ and $\{\lambda_i(\cdot),\ 1 \leq i \leq S^N(0)\}$ are both i.i.d., and mutually independent. Also $\lambda^0, \lambda \in D$, $0 \leq \lambda_1^0(t) \vee \lambda_1(t) \leq \lambda^*$, $\forall t \geq 0$ a.s. (where λ^* is a constant). We let $F^0(t) = \mathbb{P}(\eta_1^0 \leq t)$, $F(t) = \mathbb{P}(\eta_1 \leq t)$, $F_0^c(t) = 1 - F_0(t)$, $F^c(t) = 1 - F(t)$.

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The VI model 2

- We first write $S^N(t) = S^N(0) - A^N(t)$.
- The total force of infection at time t is

$$\mathfrak{F}^N(t) = \sum_{j=1}^{IN(0)} \lambda_j^0(t) + \sum_{i=1}^{A^N(t)} \lambda_i(t\tau_i^N),$$

$$A^N(t) = \int_0^t \int_0^{\bar{S}^N(s^-)\mathfrak{F}^N(s^-)} Q(ds, du)\,.$$

- Moreover

$$I^N(t) = \sum_{j=1}^{I^N(0)} 1_{\eta_j^0 > t} + \sum_{i=1}^{S^N(0)} 1_{\tau_i^N + \eta_i > t},$$

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Law of Large Numbers

- Notation : $\bar{X}^N(t) = X^N(t)/N$. As $N \to \infty$, in probability, loc. uniformly in t, $(\bar{S}^N(t), \bar{\mathfrak{F}}^N(t), \bar{I}^N(t), \bar{R}^N(t)) \to (\bar{S}(t), \bar{\mathfrak{F}}(t), \bar{I}(t), \bar{R}(t))$, where
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$$\bar{I}(t) = \bar{I}(0)F_0^c(t) + \int_0^t F^c(t-s)\bar{S}(s)\bar{\mathfrak{F}}(s)ds,$$

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- $$\bar{S}(t) = \bar{S}(0) - \int_0^t \bar{S}(s)\bar{\mathfrak{F}}(s)ds,$$ $$\bar{\mathfrak{F}}(t) = \bar{I}(0)\bar{\lambda}^0(t) + \int_0^t \bar{\lambda}(t-s)\bar{S}(s)\bar{\mathfrak{F}}(s)ds\,.$$
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Another formulation of the stochastic model

- We can rewrite

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where $A_i^N(t) = 0$ if i is susceptible at time t, $= 1$ if i has been infected by time t.

- The jump time τ_i^N of $A_i^N(t)$ is the infection time of individual i, and

$$A_i^N(t) = \int_0^t \int_0^{\bar{\mathfrak{F}}^N(s^-)} 1_{A_i^N(s^-)=0} Q_i(ds, du),$$

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Equivalence of the two formulations 1

- We want to show that $A^N(t) = \sum_{i=1}^{S^N(0)} A_i^N(t)$. Recall that $A^N(t) = \int_0^t \int_0^{S^N(s^-)\bar{\mathfrak{F}}^N(s^-)} Q(ds, du)$, while $A_i^N(t) = \int_0^t \int_0^{\bar{\mathfrak{F}}^N(s^-)} Q_i(ds, du) \wedge 1$.
- At the start of the epidemic, $\mathfrak{F}^N(s) = \sum_{j=1}^{I^N(0)} \lambda_j^0(s)$, which is independent of the Q_i's. If we define Q(s,u) by

$$\begin{cases} Q_1(s,u), & \text{if } 0 \le \frac{u}{\bar{\mathfrak{F}}^N(s)} < 1; \\ Q_2(s, u - \bar{\mathfrak{F}}^N(s)), & \text{if } 1 \le \frac{u}{\bar{\mathfrak{F}}^N(s)} < 2; \\ \cdots, & \cdots \\ Q_{S^N(0)}(s, u - (S^N(0)-1)\bar{\mathfrak{F}}^N(s)), & \text{if } S^N(0) - 1 \le \frac{u}{\bar{\mathfrak{F}}^N(s)} \le S^N(0); \\ Q_{S^N(0)+1}(s,u), & \text{if } \frac{u}{\bar{\mathfrak{F}}^N(s)} > S^N(0); \end{cases}$$

we see that $A^N(t) = \sum_{i=1}^{S^N(0)} A_i^N(t)$ until the first jump, say $\tau_{(1)}^N$. That jump of $A^N(t)$ is a jump of $A_{i_1}^N$.

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Equivalence of the two formulations 2

- Starting from that jump, we suppress the i_1 line in the above definition of Q, and for all $j > i_1$, we replace the line

$$Q_j(s, u-(j-1)\bar{\mathfrak{F}}^N(s)), \text{if } j-1 \le \frac{u}{\bar{\mathfrak{F}}^N(s)} < j \quad \text{by}$$

$$Q_j(s, u-(j-2)\bar{\mathfrak{F}}^N(s)), \qquad \text{if } j-2 \le \frac{u}{\bar{\mathfrak{F}}^N(s)} < j-1\,.$$

- Now we have $A^N(t) = \sum_{1\le i\le S^N(0),\ i\ne i_1} A_i^N(t)$, which is OK since the next jump is expected at a level $0 \le u \le (S^N(0)-1)\bar{\mathfrak{F}}^N(s) = S^N(s)\bar{\mathfrak{F}}^N(s)$ until the next jump, etc.

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Proof of the LLN 1

- Now suppose that we replace, for each $1 \le i \le S^N(0)$, $A_i^N(t)$ by $A_i(t)$ defined by

$$A_i(t) = \int_0^t \int_0^{\bar{\mathfrak{F}}(s)} 1_{A_i(s^-)=0} Q_i(ds, du) \,.$$

- We note that $\mathbb{P}(\tau_i > s) = \exp(-\int_0^s \bar{\mathfrak{F}}(r)dr)$. Recall that

$$\begin{aligned}
\bar{\mathfrak{F}}(t) &= \bar{I}(0)\bar{\lambda}^0(t) + \int_0^t \bar{\lambda}(t-s)\bar{S}(s)\bar{\mathfrak{F}}(s)ds \\
&= \bar{I}(0)\bar{\lambda}^0(t) + \bar{S}(0)\int_0^t \bar{\lambda}(t-s)\exp(-\int_0^t \bar{\mathfrak{F}}(s)ds)\bar{\mathfrak{F}}(s)ds \\
&= \bar{I}(0)\bar{\lambda}^0(t) + \bar{S}(0)\mathbb{E}[\bar{\lambda}(t-\tau_i)] \,.
\end{aligned}$$

- If we take the last line for the definition of $\bar{\mathfrak{F}}(t)$ in the formula for $A_i(t)$, we see that $A_i(t)$ solves a type of McKean-Vlasov SDE.

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- If we take the last line for the definition of $\bar{\mathfrak{F}}(t)$ in the formula for $A_i(t)$, we see that $A_i(t)$ solves a type of McKean-Vlasov SDE.

Proof of the LLN 1

- Now suppose that we replace, for each $1 \le i \le S^N(0)$, $A_i^N(t)$ by $A_i(t)$ defined by

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- The point is that the sequence $\{A_i(\cdot),\ i \geq 1\}$ is i.i.d. So by the law of large numbers in D, as $N \to \infty$, a.s., locally unif. in t,

$$\frac{1}{N}\sum_{i=1}^{S^N(0)} A_i(t) \to \bar{S}(0)\int_0^t \exp\left(-\int_0^t \bar{\mathfrak{F}}(s)ds\right)\bar{\mathfrak{F}}(s)ds$$

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Lemma

There exists a positive constant C_{T,λ^*} such that for all $N \geq 1$, $0 \leq t \leq T$,

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Time of extinction of an epidemic

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Comparison with the time of extinction in a simplified model

- Comparison of the expectations of the times of extinction :

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	$\rho = -0.0683$	$\rho = -0.03816$
Varying infectivity model	$\mathbb{E}[T_{ext}] \approx 18.7854$	$\mathbb{E}[T_{ext}] \approx 22.6568$
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