The early stage of an outbreak

Étienne Pardoux

EMA Abidjan

An epidemic does not have the branching property

- "Birth" in a branching process is "infection" in the epidemic.
- The rate at which an individual gives birth is age dependent. Before being adult this rate is 0. Then it increases and reaches a maximum, before decreasing with the age.
- So it is tempting to consider the evolution of the epidemic as a branching process. But the branching property is violated !
- Consider two infectious individuals. Their descendance are not independent, since whenever the two try to infect the same susceptible indiv. only one of the two may succeed.

An epidemic does not have the branching property

- "Birth" in a branching process is "infection" in the epidemic.
- The rate at which an individual gives birth is age dependent. Before being adult this rate is 0. Then it increases and reaches a maximum, before decreasing with the age.
- So it is tempting to consider the evolution of the epidemic as a branching process. But the branching property is violated !
- Consider two infectious individuals. Their descendance are not independent, since whenever the two try to infect the same susceptible indiv. only one of the two may succeed.

An epidemic does not have the branching property

- "Birth" in a branching process is "infection" in the epidemic.
- The rate at which an individual gives birth is age dependent. Before being adult this rate is 0. Then it increases and reaches a maximum, before decreasing with the age.
- So it is tempting to consider the evolution of the epidemic as a branching process. But the branching property is violated !
- Consider two infectious individuals. Their descendance are not independent, since whenever the two try to infect the same susceptible indiv. only one of the two may succeed.

An epidemic does not have the branching property

- "Birth" in a branching process is "infection" in the epidemic.
- The rate at which an individual gives birth is age dependent. Before being adult this rate is 0. Then it increases and reaches a maximum, before decreasing with the age.
- So it is tempting to consider the evolution of the epidemic as a branching process. But the branching property is violated !
- Consider two infectious individuals. Their descendance are not independent, since whenever the two try to infect the same susceptible indiv. only one of the two may succeed.

Large population, small # of infectious

- Suppose that the total population N is large, and that the number of infectious $k = k(N)$ is of smaller order.
- We consider the *varying infectivity* epidemic model. If $\lambda(t)$ were deterministic, the law of the # of infections generated by one infectious would be Poi($\int_0^\infty \lambda(t)dt$). Otherwise, we call that law MixPoi($\int_0^\infty \lambda(t)dt$).
- Consider the Galton Watson process with offspring distribution MixPoi($\int_0^\infty \lambda(t)dt$). The mean number of offspring is $R_0 = \int_0^\infty \mathbb{E}[\lambda(t)]dt$.
- Let $\{\lambda_i(t), t \geq 0\}_{i=0,1,\dots}$ a sequence of i.i.d. copies of $\lambda(t)$, $\{Q_i, i = 0, 1, \dots\}$ an i.i.d. sequence of standard PRMs on $\mathbb{R}_+^2$, $\{U_i, i = 1, 2, \dots\}$ an i.i.d. sequence of r.v. $\mathcal{U}([0, 1])$, the three being mutually independent.

  We also define the r.v. $\eta_i = \sup\{t,\ \lambda_i(t) > 0\}$, $i \geq 0$.

Large population, small # of infectious

- Suppose that the total population N is large, and that the number of infectious $k = k(N)$ is of smaller order.
- We consider the *varying infectivity* epidemic model. If $\lambda(t)$ were deterministic, the law of the # of infections generated by one infectious would be $\text{Poi}(\int_0^\infty \lambda(t)dt)$. Otherwise, we call that law $\text{MixPoi}(\int_0^\infty \lambda(t)dt)$.
- Consider the Galton Watson process with offspring distribution $\text{MixPoi}(\int_0^\infty \lambda(t)dt)$. The mean number of offspring is $R_0 = \int_0^\infty \mathbb{E}[\lambda(t)]dt$.
- Let $\{\lambda_i(t), t \geq 0\}_{i=0,1,\dots}$ a sequence of i.i.d. copies of $\lambda(t)$, $\{Q_i, i = 0, 1, \dots\}$ an i.i.d. sequence of standard PRMs on $\mathbb{R}_+^2$, $\{U_i, i = 1, 2, \dots\}$ an i.i.d. sequence of r.v. $\mathcal{U}([0, 1])$, the three being mutually independent.
  We also define the r.v. $\eta_i = \sup\{t,\ \lambda_i(t) > 0\}$, $i \geq 0$.

Large population, small # of infectious

- Suppose that the total population N is large, and that the number of infectious $k = k(N)$ is of smaller order.
- We consider the *varying infectivity* epidemic model. If $\lambda(t)$ were deterministic, the law of the # of infections generated by one infectious would be $\text{Poi}(\int_0^\infty \lambda(t)dt)$. Otherwise, we call that law $\text{MixPoi}(\int_0^\infty \lambda(t)dt)$.
- Consider the Galton Watson process with offspring distribution $\text{MixPoi}(\int_0^\infty \lambda(t)dt)$. The mean number of offspring is $R_0 = \int_0^\infty \mathbb{E}[\lambda(t)]dt$.
- Let $\{\lambda_i(t), t \geq 0\}_{i=0,1,\ldots}$ a sequence of i.i.d. copies of $\lambda(t)$, $\{Q_i, i = 0, 1, \ldots\}$ an i.i.d. sequence of standard PRMs on $\mathbb{R}_+^2$, $\{U_i, i = 1, 2, \ldots\}$ an i.i.d. sequence of r.v. $\mathcal{U}([0, 1])$, the three being mutually independent.
  We also define the r.v. $\eta_i = \sup\{t,\ \lambda_i(t) > 0\}$, $i \geq 0$.

Large population, small # of infectious

- Suppose that the total population N is large, and that the number of infectious $k = k(N)$ is of smaller order.
- We consider the *varying infectivity* epidemic model. If $\lambda(t)$ were deterministic, the law of the # of infections generated by one infectious would be $\text{Poi}(\int_0^\infty \lambda(t)dt)$. Otherwise, we call that law $\text{MixPoi}(\int_0^\infty \lambda(t)dt)$.
- Consider the Galton Watson process with offspring distribution $\text{MixPoi}(\int_0^\infty \lambda(t)dt)$. The mean number of offspring is $R_0 = \int_0^\infty \mathbb{E}[\lambda(t)]dt$.
- Let $\{\lambda_i(t), t \geq 0\}_{i=0,1,\dots}$ a sequence of i.i.d. copies of $\lambda(t)$, $\{Q_i, i = 0, 1, \dots\}$ an i.i.d. sequence of standard PRMs on $\mathbb{R}_+^2$, $\{U_i, i = 1, 2, \dots\}$ an i.i.d. sequence of r.v. $\mathcal{U}([0,1])$, the three being mutually independent.
  We also define the r.v. $\eta_i = \sup\{t,\ \lambda_i(t) > 0\}$, $i \geq 0$.

Exercise

Compute the offspring distribution when

$$\lambda(t) = \lambda 1_{\mathcal{L} \leq t < \mathcal{L}+\mathcal{I}}(t),$$

with $\lambda > 0$ given, in the two cases $\mathcal{I}$ deterministic, and $\mathcal{I} \sim Exp(\mu)$.

The branching process 1

- We start with an ancestor with label 0, to which we attach the pair (λ_0, Q_0). The ancestor gives birth at the jump times of the process

$$\int_0^t \int_0^\infty 1_{u \le \lambda_0(s)} Q_0(ds, du),$$

  which we label $\tau_{0,1}, \tau_{0,2}, \ldots$.
- The first born indiv. at time $T_1 = \tau_{0,1}$ is given the label 1 and attached (λ_1, Q_1), which describe the descent of indiv. 1, with births at times $T_1 + \tau_{1,1}, T_1 + \tau_{1,2}, \ldots$.
- The second born individual is born at time $T_2 = \inf\{\tau_{0,2}, T_1 + \tau_{1,1}\}$, to which is attached (λ_2, Q_2), hence the descent at times $T_2 + \tau_{2,1}, T_2 + \tau_{2,2}, \ldots$.
- The third born individual is born at time

$$T_3 = \begin{cases} \inf\{\tau_{0,3}, T_1 + \tau_{1,1}, T_2 + \tau_{2,1}\}, & \text{if } T_2 = \tau_{0,2}, \\ \inf\{\tau_{0,2}, T_1 + \tau_{1,2}, T_2 + \tau_{2,1}\}, & \text{if } T_2 = T_1 + \tau_{1,1}. \end{cases}$$

etc.

The branching process 1

- We start with an ancestor with label 0, to which we attach the pair (λ_0, Q_0). The ancestor gives birth at the jump times of the process

$$\int_0^t \int_0^\infty 1_{u \le \lambda_0(s)} Q_0(ds, du),$$

which we label $\tau_{0,1}, \tau_{0,2}, \ldots$.

- The first born indiv. at time $T_1 = \tau_{0,1}$ is given the label 1 and attached (λ_1, Q_1), which describe the descent of indiv. 1, with births at times $T_1 + \tau_{1,1}$, $T_1 + \tau_{1,2}, \ldots$.
- The second born individual is born at time $T_2 = \inf\{\tau_{0,2}, T_1 + \tau_{1,1}\}$, to which is attached (λ_2, Q_2), hence the descent at times $T_2 + \tau_{2,1}$, $T_2 + \tau_{2,2}, \ldots$.
- The third born individual is born at time

$$T_3 = \begin{cases} \inf\{\tau_{0,3}, T_1 + \tau_{1,1}, T_2 + \tau_{2,1}\}, & \text{if } T_2 = \tau_{0,2}, \\ \inf\{\tau_{0,2}, T_1 + \tau_{1,2}, T_2 + \tau_{2,1}\}, & \text{if } T_2 = T_1 + \tau_{1,1}. \end{cases}$$

etc.

The branching process 1

- We start with an ancestor with label 0, to which we attach the pair (λ_0, Q_0). The ancestor gives birth at the jump times of the process

$$\int_0^t \int_0^\infty 1_{u \le \lambda_0(s)} Q_0(ds, du),$$

which we label $\tau_{0,1}, \tau_{0,2}, \ldots$.

- The first born indiv. at time $T_1 = \tau_{0,1}$ is given the label 1 and attached (λ_1, Q_1), which describe the descent of indiv. 1, with births at times $T_1 + \tau_{1,1}, T_1 + \tau_{1,2}, \ldots$.
- The second born individual is born at time $T_2 = \inf\{\tau_{0,2}, T_1 + \tau_{1,1}\}$, to which is attached (λ_2, Q_2), hence the descent at times $T_2 + \tau_{2,1}, T_2 + \tau_{2,2}, \ldots$.
- The third born individual is born at time

$$T_3 = \begin{cases} \inf\{\tau_{0,3}, T_1 + \tau_{1,1}, T_2 + \tau_{2,1}\}, & \text{if } T_2 = \tau_{0,2}, \\ \inf\{\tau_{0,2}, T_1 + \tau_{1,2}, T_2 + \tau_{2,1}\}, & \text{if } T_2 = T_1 + \tau_{1,1}. \end{cases}$$

etc.

The branching process 1

- We start with an ancestor with label 0, to which we attach the pair (λ_0, Q_0). The ancestor gives birth at the jump times of the process

$$\int_0^t \int_0^\infty 1_{u \le \lambda_0(s)} Q_0(ds, du)\,,$$

which we label $\tau_{0,1}, \tau_{0,2}, \ldots$.
- The first born indiv. at time $T_1 = \tau_{0,1}$ is given the label 1 and attached (λ_1, Q_1), which describe the descent of indiv. 1, with births at times $T_1 + \tau_{1,1}$, $T_1 + \tau_{1,2}, \ldots$.
- The second born individual is born at time $T_2 = \inf\{\tau_{0,2}, T_1 + \tau_{1,1}\}$, to which is attached (λ_2, Q_2), hence the descent at times $T_2 + \tau_{2,1}$, $T_2 + \tau_{2,2}, \ldots$.
- The third born individual is born at time

$$T_3 = \begin{cases} \inf\{\tau_{0,3}, T_1 + \tau_{1,1}, T_2 + \tau_{2,1}\}, & \text{if } T_2 = \tau_{0,2}, \\ \inf\{\tau_{0,2}, T_1 + \tau_{1,2}, T_2 + \tau_{2,1}\}, & \text{if } T_2 = T_1 + \tau_{1,1}\,. \end{cases}$$

etc.

The branching process 2

- We now define (with $T_0 = 0$) for $t > 0$:

$$I(t) = \sum_{i \geq 0} 1_{T_i \leq t < T_i + \eta_i},$$
$$R(t) = \sum_{i \geq 0} 1_{T_i + \eta_i \leq t} \,.$$

- Note that those processes do not depend upon N !

The branching process 2

- We now define (with $T_0 = 0$) for $t > 0$:

$$I(t) = \sum_{i \geq 0} 1_{T_i \leq t < T_i + \eta_i},$$
$$R(t) = \sum_{i \geq 0} 1_{T_i + \eta_i \leq t} \,.$$

- Note that those processes do not depend upon N !

The epidemic model with N initially susceptibles 1

- Initially we have N susceptibles $+$ one indiv. ($=$ the ancestor of the BP) who is infected at time 0, to which is attached (λ_0, Q_0). The first infectious contact is with the individual $[U_1 N] + 1$. This is individual 1, to which is attached (λ_1, Q_1).
- The second potential infection hits individual $[U_2 N] + 1$. If $[U_2 N] + 1 = [U_1 N] + 1$, then nothing happens : we say that a *ghost* has been infected. Otherwise, $[U_2 N] + 1$ is the second infected individual, to which is attached (λ_2, Q_2) as in the branching process.
- We go on. When the contact is with an already infected individual, we say that a *ghost* appears, and the corresponding descent in the branching process is suppressed.

The epidemic model with N initially susceptibles 1

- Initially we have N susceptibles $+$ one indiv. ($=$ the ancestor of the BP) who is infected at time 0, to which is attached (λ_0, Q_0). The first infectious contact is with the individual $[U_1 N] + 1$. This is individual 1, to which is attached (λ_1, Q_1).
- The second potential infection hits individual $[U_2 N] + 1$. If $[U_2 N] + 1 = [U_1 N] + 1$, then nothing happens : we say that a *ghost* has been infected. Otherwise, $[U_2 N] + 1$ is the second infected individual, to which is attached (λ_2, Q_2) as in the branching process.
- We go on. When the contact is with an already infected individual, we say that a *ghost* appears, and the corresponding descent in the branching process is suppressed.

The epidemic model with N initially susceptibles 1

- Initially we have N susceptibles + one indiv. (= the ancestor of the BP) who is infected at time 0, to which is attached (λ_0, Q_0). The first infectious contact is with the individual $[U_1 N] + 1$. This is individual 1, to which is attached (λ_1, Q_1).
- The second potential infection hits individual $[U_2 N] + 1$. If $[U_2 N] + 1 = [U_1 N] + 1$, then nothing happens : we say that a *ghost* has been infected. Otherwise, $[U_2 N] + 1$ is the second infected individual, to which is attached (λ_2, Q_2) as in the branching process.
- We go on. When the contact is with an already infected individual, we say that a *ghost* appears, and the corresponding descent in the branching process is suppressed.

The epidemic model with N initially susceptibles 2

- Let $\{i_j,\ j = 0, 1, 2, \ldots\}$ denote the strictly increasing sequence starting from $i_0 = 0$, $i_1 = 1$ and such that for $j \geq 2$, $[U_{i_j}N] + 1 \notin \cup_{1 \leq k < i_j}\{[U_k N] + 1\}$, i.e. the i_j–th birth in the BP does not create a ghost in the N–epidemic.
- At time 0, we have $(S^N(0), I^N(0), R^N(0)) = (N, 1, 0)$. At each time T_{i_j}, $j \geq 1$, S^N decreases by 1 and I^N increases by 1. At each time $T_{i_j} + \eta_{i_j}$, $j \geq 0$, I^N decreases by 1 and R^N increases by 1. Recall that $\eta_{i_j} = \sup\{t > 0,\ \lambda_{i_j}(t) > 0\}$.
- The epidemic goes on until $I^N(t) = 0$. The final size equals the value of $R^N(t)$ at that time. Note that for all times, $S^N(t) + I^N(t) + R^N(t) = N + 1$.
- We have constructed the BP and the N–epidemic for all values of N jointly on the same probability space.

The epidemic model with N initially susceptibles 2

- Let $\{i_j,\ j=0,1,2,\ldots\}$ denote the strictly increasing sequence starting from $i_0=0$, $i_1=1$ and such that for $j\ge 2$, $[U_{i_j}N]+1 \not\in \cup_{1\le k<i_j}\{[U_kN]+1\}$, i.e. the i_j–th birth in the BP does not create a ghost in the N–epidemic.
- At time 0, we have $(S^N(0), I^N(0), R^N(0)) = (N,1,0)$. At each time T_{i_j}, $j\ge 1$, S^N decreases by 1 and I^N increases by 1. At each time $T_{i_j}+\eta_{i_j}$, $j\ge 0$, I^N decreases by 1 and R^N increases by 1. Recall that $\eta_{i_j}=\sup\{t>0,\ \lambda_{i_j}(t)>0\}$.
- The epidemic goes on until $I^N(t)=0$. The final size equals the value of $R^N(t)$ at that time. Note that for all times, $S^N(t)+I^N(t)+R^N(t)=N+1$.
- We have constructed the BP and the N–epidemic for all values of N jointly on the same probability space.

The epidemic model with N initially susceptibles 2

- Let $\{i_j,\ j = 0, 1, 2, \ldots\}$ denote the strictly increasing sequence starting from $i_0 = 0$, $i_1 = 1$ and such that for $j \geq 2$, $[U_{i_j} N] + 1 \notin \cup_{1 \leq k < i_j} \{[U_k N] + 1\}$, i.e. the i_j–th birth in the BP does not create a ghost in the N–epidemic.
- At time 0, we have $(S^N(0), I^N(0), R^N(0)) = (N, 1, 0)$. At each time T_{i_j}, $j \geq 1$, S^N decreases by 1 and I^N increases by 1. At each time $T_{i_j} + \eta_{i_j}$, $j \geq 0$, I^N decreases by 1 and R^N increases by 1. Recall that $\eta_{i_j} = \sup\{t > 0,\ \lambda_{i_j}(t) > 0\}$.
- The epidemic goes on until $I^N(t) = 0$. The final size equals the value of $R^N(t)$ at that time. Note that for all times, $S^N(t) + I^N(t) + R^N(t) = N + 1$.
- We have constructed the BP and the N–epidemic for all values of N jointly on the same probability space.

The epidemic model with N initially susceptibles 2

- Let $\{i_j,\ j = 0, 1, 2, \ldots\}$ denote the strictly increasing sequence starting from $i_0 = 0$, $i_1 = 1$ and such that for $j \geq 2$, $[U_{i_j} N] + 1 \notin \cup_{1 \leq k < i_j} \{[U_k N] + 1\}$, i.e. the i_j–th birth in the BP does not create a ghost in the N–epidemic.
- At time 0, we have $(S^N(0), I^N(0), R^N(0)) = (N, 1, 0)$. At each time T_{i_j}, $j \geq 1$, S^N decreases by 1 and I^N increases by 1. At each time $T_{i_j} + \eta_{i_j}$, $j \geq 0$, I^N decreases by 1 and R^N increases by 1. Recall that $\eta_{i_j} = \sup\{t > 0,\ \lambda_{i_j}(t) > 0\}$.
- The epidemic goes on until $I^N(t) = 0$. The final size equals the value of $R^N(t)$ at that time. Note that for all times, $S^N(t) + I^N(t) + R^N(t) = N + 1$.
- We have constructed the BP and the N–epidemic for all values of N jointly on the same probability space.

The early stage 1

- The epidemic and the BP coincide until T^N = the time of appearance of the first ghost ($= +\infty$ in case no ghost appears). Let M^N denote the # of infections prior to the first ghost. We have

Theorem

For all $t \in [0, T^N)$, $(I^N(t), R^N(t)) = (I(t), R(t))$.
Moreover, $T^N \to \infty$ in probability, as $N \to \infty$. The same is true for M^N, unless the BP is (sub)critical, in which case $\mathbb{P}(T^N = +\infty) \to 1$.

- **Proof** The first statement is obvious. To prove the second statement, we first compute

$$\mathbb{P}(M^N > k) = 1 \times \frac{N-1}{N} \times \cdots \times \frac{N-k}{N} = \prod_{j=0}^{k}\left(1 - \frac{j}{N}\right)$$

$$\geq 1 - \sum_{j=1}^{k} \frac{j}{N} = 1 - \frac{k(k+1)}{2N}.$$

The early stage 1

- The epidemic and the BP coincide until T^N = the time of appearance of the first ghost ($= +\infty$ in case no ghost appears). Let M^N denote the # of infections prior to the first ghost. We have

Theorem

For all $t \in [0, T^N)$, $(I^N(t), R^N(t)) = (I(t), R(t))$.
Moreover, $T^N \to \infty$ in probability, as $N \to \infty$. The same is true for M^N, unless the BP is (sub)critical, in which case $\mathbb{P}(T^N = +\infty) \to 1$.

- **Proof** The first statement is obvious. To prove the second statement, we first compute

$$\mathbb{P}(M^N > k) = 1 \times \frac{N-1}{N} \times \cdots \times \frac{N-k}{N} = \prod_{j=0}^{k} \left(1 - \frac{j}{N}\right)$$

$$\geq 1 - \sum_{j=1}^{k} \frac{j}{N} = 1 - \frac{k(k+1)}{2N}.$$

The early stage 2. End proof of Theorem

- We have $\mathbb{P}(M^N > k) \geq 1 - \frac{k(k+1)}{2N}$. So if $k(N) = o(\sqrt{N})$ (e.g. $k(N) = N^{1/3}$, $k(N) = \log(N)$), then $\mathbb{P}(M^N > k(N)) \to 1$, as $N \to \infty$.
- Let $Z(t)$ be the # of indiv. born on $[0, t]$ in the BP, and $Z^N(t) = N - S^N(t)$. We have $Z^N(t) = Z(t)$ for $t < T^N$. Since $\mathbb{P}(M^N > k(N)) \to 1$, $\mathbb{P}(\inf\{t; Z(t) = k(N)\} \leq T^N) \to 1$.
- If the BP is (sub)critical, then $Z(t)$ remains bounded, and $\mathbb{P}(T^N = +\infty) \to 1$.
- In the supercritical case, $\exists \rho > 0$ s.t. $\int_0^\infty \bar{\lambda}(t) e^{-\rho t} dt = 1$, $Z(t) \sim W e^{\rho t}$, with $\{W = 0\} = Ext$. Now, with $W' > W$, $\mathbb{P}(k(N) \leq W' e^{\rho T^N}) \to 1$, hence $\mathbb{P}(T^N \geq \frac{\log k(N) - \log W'}{\rho}) \to 1$, and $T^N \to \infty$ in probability.

The early stage 2. End proof of Theorem

- We have $\mathbb{P}(M^N > k) \geq 1 - \frac{k(k+1)}{2N}$. So if $k(N) = o(\sqrt{N})$ (e.g. $k(N) = N^{1/3}$, $k(N) = \log(N)$), then $\mathbb{P}(M^N > k(N)) \to 1$, as $N \to \infty$.
- Let $Z(t)$ be the # of indiv. born on $[0, t]$ in the BP, and $Z^N(t) = N - S^N(t)$. We have $Z^N(t) = Z(t)$ for $t < T^N$. Since $\mathbb{P}(M^N > k(N)) \to 1$, $\mathbb{P}(\inf\{t; Z(t) = k(N)\} \leq T^N) \to 1$.
- If the BP is (sub)critical, then $Z(t)$ remains bounded, and $\mathbb{P}(T^N = +\infty) \to 1$.
- In the supercritical case, $\exists \rho > 0$ s.t. $\int_0^\infty \bar{\lambda}(t) e^{-\rho t} dt = 1$, $Z(t) \sim W e^{\rho t}$, with $\{W = 0\} = Ext$. Now, with $W' > W$, $\mathbb{P}(k(N) \leq W' e^{\rho T^N}) \to 1$, hence $\mathbb{P}(T^N \geq \frac{\log k(N) - \log W'}{\rho}) \to 1$, and $T^N \to \infty$ in probability.

The early stage 2. End proof of Theorem

- We have $\mathbb{P}(M^N > k) \geq 1 - \frac{k(k+1)}{2N}$. So if $k(N) = o(\sqrt{N})$ (e.g. $k(N) = N^{1/3}$, $k(N) = \log(N)$), then $\mathbb{P}(M^N > k(N)) \to 1$, as $N \to \infty$.
- Let $Z(t)$ be the # of indiv. born on $[0, t]$ in the BP, and $Z^N(t) = N - S^N(t)$. We have $Z^N(t) = Z(t)$ for $t < T^N$. Since $\mathbb{P}(M^N > k(N)) \to 1$, $\mathbb{P}(\inf\{t; Z(t) = k(N)\} \leq T^N) \to 1$.
- If the BP is (sub)critical, then $Z(t)$ remains bounded, and $\mathbb{P}(T^N = +\infty) \to 1$.
- In the supercritical case, $\exists \rho > 0$ s.t. $\int_0^\infty \bar{\lambda}(t) e^{-\rho t} dt = 1$, $Z(t) \sim W e^{\rho t}$, with $\{W = 0\} = Ext$. Now, with $W' > W$, $\mathbb{P}(k(N) \leq W' e^{\rho T^N}) \to 1$, hence $\mathbb{P}(T^N \geq \frac{\log k(N) - \log W'}{\rho}) \to 1$, and $T^N \to \infty$ in probability.

The early stage 2. End proof of Theorem

- We have $\mathbb{P}(M^N > k) \geq 1 - \frac{k(k+1)}{2N}$. So if $k(N) = o(\sqrt{N})$ (e.g. $k(N) = N^{1/3}$, $k(N) = \log(N)$), then $\mathbb{P}(M^N > k(N)) \to 1$, as $N \to \infty$.
- Let $Z(t)$ be the # of indiv. born on $[0, t]$ in the BP, and $Z^N(t) = N - S^N(t)$. We have $Z^N(t) = Z(t)$ for $t < T^N$. Since $\mathbb{P}(M^N > k(N)) \to 1$, $\mathbb{P}(\inf\{t; Z(t) = k(N)\} \leq T^N) \to 1$.
- If the BP is (sub)critical, then $Z(t)$ remains bounded, and $\mathbb{P}(T^N = +\infty) \to 1$.
- In the supercritical case, $\exists \rho > 0$ s.t. $\int_0^\infty \bar{\lambda}(t) e^{-\rho t} dt = 1$, $Z(t) \sim W e^{\rho t}$, with $\{W = 0\} = Ext$. Now, with $W' > W$, $\mathbb{P}(k(N) \leq W' e^{\rho T^N}) \to 1$, hence $\mathbb{P}(T^N \geq \frac{\log k(N) - \log W'}{\rho}) \to 1$, and $T^N \to \infty$ in probability.

The early stage 3

- Define $Z = Z(\infty)$, $Z^N = Z^N(\infty)$.
- We have first :

Corollary

If $R_0 \leq 1$, then with probability converging to 1 as $N \to \infty$, $(I^N(t), R^N(t)) = (I(t), R(t))$ for all $t \geq 0$. And $\mathbb{P}(Z^N = k) \to \mathbb{P}(Z = k)$ for all $k \geq 1$.

- We have next :

Corollary

If $R_0 > 1$, then $\mathbb{P}(Z^N = k) \to \mathbb{P}(Z = k)$ for all $k \geq 1$. Moreover

$$\mathbb{P}(\lim_N Z^N = +\infty) = \mathbb{P}(Z = +\infty) = 1 - \mathbb{P}(Ext)\,.$$

- A BP approximation has been used for modeling the spread of EBOLA during an early stage in West Africa in 2014.

The early stage 3

- Define $Z = Z(\infty)$, $Z^N = Z^N(\infty)$.
- We have first :

Corollary

If $R_0 \leq 1$, then with probability converging to 1 as $N \to \infty$, $(I^N(t), R^N(t)) = (I(t), R(t))$ for all $t \geq 0$. And $\mathbb{P}(Z^N = k) \to \mathbb{P}(Z = k)$ for all $k \geq 1$.

- We have next :

Corollary

If $R_0 > 1$, then $\mathbb{P}(Z^N = k) \to \mathbb{P}(Z = k)$ for all $k \geq 1$. Moreover

$$\mathbb{P}(\lim_N Z^N = +\infty) = \mathbb{P}(Z = +\infty) = 1 - \mathbb{P}(Ext).$$

- A BP approximation has been used for modeling the spread of EBOLA during an early stage in West Africa in 2014.

The early stage 3

- Define $Z = Z(\infty)$, $Z^N = Z^N(\infty)$.
- We have first :

Corollary

If $R_0 \leq 1$, then with probability converging to 1 as $N \to \infty$, $(I^N(t), R^N(t)) = (I(t), R(t))$ for all $t \geq 0$. And $\mathbb{P}(Z^N = k) \to \mathbb{P}(Z = k)$ for all $k \geq 1$.

- We have next :

Corollary

If $R_0 > 1$, then $\mathbb{P}(Z^N = k) \to \mathbb{P}(Z = k)$ for all $k \geq 1$. Moreover

$$\mathbb{P}(\lim_N Z^N = +\infty) = \mathbb{P}(Z = +\infty) = 1 - \mathbb{P}(Ext) \,.$$

- A BP approximation has been used for modeling the spread of EBOLA during an early stage in West Africa in 2014.

The early stage 3

- Define $Z = Z(\infty)$, $Z^N = Z^N(\infty)$.
- We have first :

Corollary

If $R_0 \leq 1$, then with probability converging to 1 as $N \to \infty$, $(I^N(t), R^N(t)) = (I(t), R(t))$ for all $t \geq 0$. And $\mathbb{P}(Z^N = k) \to \mathbb{P}(Z = k)$ for all $k \geq 1$.

- We have next :

Corollary

If $R_0 > 1$, then $\mathbb{P}(Z^N = k) \to \mathbb{P}(Z = k)$ for all $k \geq 1$. Moreover

$$\mathbb{P}(\lim_N Z^N = +\infty) = \mathbb{P}(Z = +\infty) = 1 - \mathbb{P}(Ext)\,.$$

- A BP approximation has been used for modeling the spread of EBOLA during an early stage in West Africa in 2014.

Probability of a major / minor epidemic

- If $R_0 \leq 1$, then the BP goes extinct in finite time a.s. The N epidemic is upper bounded by the BP for all N, so there is no major outbreak, the proportion of the total number of individuals getting infected tends to 0 as N tends to 0.
- If $R_0 > 1$, with the probability $p_{Ext} \in (0,1)$, the BP goes extinct in finite time (in which case there is no major outbreak). With probability $1 - p_{Ext}$, the BP goes off at exponential rate, and there is a major epidemic. How can we compute p_{Ext} ?
- p_{Ext} is also the probability of extinction of the discrete time BP, i.e. it is the solution of the equation $g(s) = s$, if g is the generating function of the number of offspring, which follows the MixPoi$(\int_0^\infty \lambda(t)dt)$ distribution. If $X \sim$ Poi(a), then $\mathbb{E}[s^X] = \exp([s-1]a)$. So $g(s) = \mathbb{E}\left\{\exp\left([s-1]\int_0^\infty \lambda(t)dt\right)\right\}$.

Probability of a major / minor epidemic

- If $R_0 \leq 1$, then the BP goes extinct in finite time a.s. The N epidemic is upper bounded by the BP for all N, so there is no major outbreak, the proportion of the total number of individuals getting infected tends to 0 as N tends to 0.
- If $R_0 > 1$, with the probability $p_{Ext} \in (0, 1)$, the BP goes extinct in finite time (in which case there is no major outbreak). With probability $1 - p_{Ext}$, the BP goes off at exponential rate, and there is a major epidemic. How can we compute p_{Ext} ?
- p_{Ext} is also the probability of extinction of the discrete time BP, i.e. it is the solution of the equation $g(s) = s$, if g is the generating function of the number of offspring, which follows the $\text{MixPoi}(\int_0^\infty \lambda(t)dt)$ distribution. If $X \sim \text{Poi}(a)$, then $\mathbb{E}[s^X] = \exp([s-1]a)$. So $g(s) = \mathbb{E}\left\{\exp\left([s-1]\int_0^\infty \lambda(t)dt\right)\right\}$.

Probability of a major / minor epidemic

- If $R_0 \leq 1$, then the BP goes extinct in finite time a.s. The N epidemic is upper bounded by the BP for all N, so there is no major outbreak, the proportion of the total number of individuals getting infected tends to 0 as N tends to 0.
- If $R_0 > 1$, with the probability $p_{Ext} \in (0, 1)$, the BP goes extinct in finite time (in which case there is no major outbreak). With probability $1 - p_{Ext}$, the BP goes off at exponential rate, and there is a major epidemic. How can we compute p_{Ext} ?
- p_{Ext} is also the probability of extinction of the discrete time BP, i.e. it is the solution of the equation $g(s) = s$, if g is the generating function of the number of offspring, which follows the $\text{MixPoi}(\int_0^\infty \lambda(t)dt)$ distribution. If $X \sim \text{Poi}(a)$, then $\mathbb{E}[s^X] = \exp([s-1]a)$. So $g(s) = \mathbb{E}\left\{\exp\left([s-1]\int_0^\infty \lambda(t)dt\right)\right\}$.

Complements

Exercise Compute $g(s)$ in the case of the Markov model, i.e. $\lambda(t) = \lambda 1_{t \le \mathcal{I}}$, and $\mathcal{I}$ follows the Exp(μ) distribution. Compute p_{Ext} in that case.

Remark p_{Ext} does not depend only upon $R_0 = \int_0^\infty \bar{\lambda}(t)dt$!