Basics of Epidemic Models

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The compartments

- Most of the coming lectures will study epidemic compartment SIR / SEIR models, Here
  - S stands for Susceptible, $S(t) = \#$ of susceptible indiv. at time t;
  - E for Exposed, $E(t) = \#$ of exposed indiv. at time t;
  - I for Infectious, $I(t) = \#$ of infectious indiv. at time t;
  - R for Recovered, $R(t) = \#$ of recovered indiv. at time t.
- We shall mainly consider no demography. Hence if $N = S(0) + E(0) + I(0) + R(0)$, $S(t) + E(t) + I(t) + R(t) = N$ for all $t > 0$.
- A susceptible who gets infected is first E, then I, then R, at which point he/she is immune, and in most of the lectures we shall assume that he/she remains immune for ever.
- However, we shall also consider SIS / SEIS and SIRS / SEIRS models, where infectious either become susceptible as soon as they cure, or lose their immunity after some time.

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Evolution of the epidemic

- While I, an individual has contacts with others at rate β. The contacted individual is chosen uniformly at random in the population. If the contacted individual is S, then the contact results in a new infection with probability p. Otherwise, nothing happens.
- In other words, an I has (potentially) infectious contacts at rate $\lambda = \beta \times p$. Such a contact results at time t in a new infection with probability $\bar{S}^N(t) = S(t)/N$.
- After being infected, an individual is first E for a random duration $\mathcal{L}$, then I for a random duration $\mathcal{I}$, after which he/she recovers.
- "At rate λ" means according to a rate λ Poisson process. We always assume that the various Poisson processes are mutually independent, and independent of the $(\mathcal{L},\mathcal{I})$ attached to the various indiv. (those are i.i.d.).
- An assumption which is not realistic, but makes the analysis easier is that the r.v. $\mathcal{L}$ and $\mathcal{I}$ are independent and both follow an exponential distribution.

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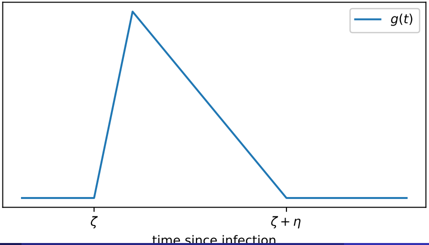
The basic reproduction number R_0

- In our homogeneous model, R_0 is the mean number of infections produced by one I *at the start of the epidemic*, i.e. while $\bar{S}^N(t) = S(t)/N \simeq 1$.
- In the above described model, $R_0 = \lambda\mathbb{E}[\mathcal{I}]$. If the law of $\mathcal{I}$ is $\text{Exp}(\mu)$, then $R_0 = \lambda/\mu$.
- **Generalisation : VI** Suppose now that the infectivity is a random function $\lambda(t)$ of the time since infection, those random functions being i.i.d. for the various individuals. Then $R_0 = \int_0^\infty \mathbb{E}[\lambda(t)]dt$. Note that in the previous case $\lambda(t) = \lambda 1_{\mathcal{I}>t}$.
  Exercise : The two formulas are consistent !



$R_0 \leq 1$ vs. $R_0 > 1$

- If $R_0 < 1$, then in average each infectious infects less than one individual. The number of infected individuals tend to decrease. If we start with a small number of infected, there will be no major epidemic : the total number of infected will be small.
- It follows from the results on branching processes that the same is true in case $R_0 = 1$.
- Ir $R_0 > 1$ however, in average the number of infected individuals increases. As a result, starting from a small number of infected individuals, there is a positive probability that a major epidemic develops.

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The escape probability

- It follows from the above description that the probability that one given S escapes infection from one particular infectious individual is

$$\mathbb{E}\left[\exp\left(-\lambda\mathcal{I}/N\right)\right] .$$

- or in the VI model

$$\mathbb{E}\left[\exp\left(-N^{-1}\int_0^\infty \lambda(t)dt\right)\right] .$$

- **Exercise** Consider a SIR / SEIR epidemic with $\lambda = 1.8$ and $\mathcal{I}$ follows the Exp(1) distribution, in a village of size $N = 100$. What is the probability that a given S escapes the infection by a given I.

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The final size 1

- Let us now deduce a famous formula by an intuitive argument (to be justified later). Consider a SIR / SEIR epidemic in a population of size N with N large. The epidemic ends soon or later. What is the proportion $\bar{\mathrm{I}}^N$ of the population which gets infected at some stage ?
- The probability of escaping the infection from all infected equals

$$\left\{\mathbb{E}\left[\exp\left(-N^{-1}\int_0^\infty \lambda(t)dt\right)\right]\right\}^{N\bar{\mathrm{I}}^N}.$$

- That probability should equal $1-\bar{\mathrm{I}}^N$. Hence

$$\begin{aligned} 1-\bar{\mathrm{I}}^N &= \left\{\mathbb{E}\left[\exp\left(-N^{-1}\int_0^\infty \lambda(t)dt\right)\right]\right\}^{N\bar{\mathrm{I}}^N} \\ &\simeq \left\{1-N^{-1}\int_0^\infty \mathbb{E}[\lambda(t)]dt\right\}^{N\bar{\mathrm{I}}^N} \end{aligned}$$

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The final size 2

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$$\log(1-\bar{\mathrm{I}}^N) \simeq -\bar{\mathrm{I}}^N \int_0^\infty \mathbb{E}[\lambda(t)]dt,$$

$$1-\bar{\mathrm{I}}^N \simeq \exp\left(-\bar{\mathrm{I}}^N \int_0^\infty \mathbb{E}[\lambda(t)]dt\right).$$

- We thus expect to have in the limit $N \to \infty$

$$1-\bar{\mathrm{I}} = \exp(-R_0\bar{\mathrm{I}}).$$

- Two remarks :
  - No all the population gets infected.
  - The final size depends only upon R_0, and not upon further details of the model.
  - The final size is zero for $R_0 = 1$, and increases with R_0.

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The SIR ODE model

Consider the SIR model in the Markovian case. One of the results which we will show in these lectures is that as $N \to \infty$, $N^{-1}(\mathsf{S}^N(t), \mathsf{I}^N(t), \mathsf{R}^N(t)) \to (\bar{\mathsf{S}}, \bar{\mathsf{I}}, \bar{\mathsf{R}})$, where $(\bar{\mathsf{S}}, \bar{\mathsf{I}}, \bar{\mathsf{R}})$ solves the ODE

$$\frac{d\bar{\mathsf{S}}}{dt}(t) = -\lambda \bar{\mathsf{S}}(t)\bar{\mathsf{I}}(t),$$
$$\frac{d\bar{\mathsf{I}}}{dt}(t) = \lambda \bar{\mathsf{S}}(t)\bar{\mathsf{I}}(t) - \mu \bar{\mathsf{I}}(t),$$
$$\frac{d\bar{\mathsf{R}}}{dt}(t) = \mu \bar{\mathsf{I}}(t)\,.$$

λ = contact rate, μ = rate of recovery.

The start of the epidemic

- At the start of the epidemic, $\bar{\mathsf{S}}(t) \simeq 1$, so we can linearize the second equation, yielding, after multiplication by N,

$$\frac{d\mathsf{I}^N}{dt}(t) = (\lambda - \mu)\mathsf{I}^N(t),$$

whose solution is

$$\mathsf{I}^N(t) = \mathsf{I}^N(0)\exp[(\lambda - \mu)t] \,.$$

- $R_0 > 1 \iff \rho := \lambda - \mu > 0$
- ρ is the growth rate. It is easy to estimate in practice (easily deduced from the doubling time = number of days necessary for the number of infected to double). What additional information is necessary for estimating R_0 once ρ is known ?
- In the VI case, ρ is such that $\int_0^\infty \mathbb{E}[\lambda(t)]e^{-\rho t}dt = 1$, i.e., if we define $g(t) := R_0^{-1}\mathbb{E}[\lambda(t)]$, then ρ and R_0 are related by $\int_0^\infty g(t)e^{-\rho t}dt = R_0^{-1}$. $g(t)$ is the "generation time distribution". **Exercise** Verify the consistency with the above Markov/ODE model.

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